Filtrations + associated graded rings + modules

In this section, we give some CA defs that allow us to construct some important geom. objects such as the blowup + tangent cone.

Def: A multiplicative filtration of a ring R is a sequence of ideals

$$R = I_0 \supseteq I_1 \supseteq \dots \quad \text{s.t.} \quad I_i I_j \subseteq I_{i+j} \text{ for all } i,j.$$

Ex: If $I \subseteq R$ any ideal, then

$$R \supseteq I \supseteq I^2 \supseteq I^3 \supseteq \dots \quad \text{is a filtration,}$$

called the I-adic filtration.

We can generalize this to modules: $M \supseteq IM \supseteq I^2 M \supseteq \dots$ is the I-adic filtration of the R-module M. Even more generally:

Def: M an R-module. A chain of submodules

$$M = M_0 \supset M_1 \supset \dots$$

is an I-filtration if $IM_n \subseteq M_{n+1}$ $\forall$ $n \geq 0$.

An I-filtration is I-stable if $IM_n = M_{n+1}$ for $n >> 0$.

Associated graded rings + modules

Let $I \subseteq R$ an ideal

Def: The <u>associated graded ring of R w.r.t. I</u> is

$$gr_I R := R/I \oplus I/I^2 \oplus \cdots$$

w/ multiplication as follows: if $\bar{a} \in I^m/I^{m+1}$, $\bar{b} \in I^n/I^{n+1}$

s.t. $a \in I^m$, $b \in I^n$, then $ab \in I^{m+n}$. Define

$$\bar{a} \cdot \bar{b} = \overline{ab} \in I^{m+n}/I^{m+n+1}$$

Why is this well-defined?

If $\overline{a'} = \bar{a}$ and $\overline{b'} = \bar{b}$ in I^m/I^{m+1} and I^n/I^{n+1}, resp.,

then $a' = a + x$, $b' = b + y$ for some $x \in I^{m+1}$, $y \in I^{n+1}$.

$$\text{Thus, } a'b' = ab + \overbrace{\underbrace{ay + bx}_{\text{in } I^{m+n+1}} + \underbrace{xy}_{\text{in } I^{m+n+2}}}^{\text{in } I^{m+n+1}}$$

$$\Rightarrow \overline{a'b'} = \overline{ab} \text{ in } I^{m+n}/I^{m+n+1}.$$

More generally, if

$$\mathcal{J}: \quad M = M_0 \supseteq M_1 \supseteq \cdots$$

is an I-filtration of M, an R-module, define

$$gr_{\mathcal{Y}} M := M/M_1 \oplus M_1/M_2 \oplus \cdots$$

This is a $gr_I R$-module as follows: If $\bar{a} \in I^m/I^{m+1}$, $\bar{b} \in M_n/M_{n+1}$, then $ab \in I^m M_n \subseteq M_{n+m}$. Set

$$\bar{a} \cdot \bar{b} = \overline{ab} \in M_{n+m}/M_{n+m+1}.$$

Stability of a filtration is important:

Prop: Let $I \subseteq R$ be an ideal and M a finitely generated R-module. If

$$\mathcal{Y}: M = M_0 \supseteq M_1 \supseteq \cdots$$

is an I-stable filtration, w/ M_i f.g. for all i, then $gr_{\mathcal{Y}} M$ is a finitely generated $gr_I R$-module.

Pf: Stability $\Rightarrow$ we can find n s.t. $IM_i = M_{i+1}$ for $i \geq n$. So for $i \geq n$, we have

$$\left(I/I^2\right)\left(M_i/M_{i+1}\right) \subseteq M_{i+1}/M_{i+2} = IM_i/I^2M_i.$$

Let $\overline{rm} \in IM_i/I^2M_i$ w/ $r \in I$, $m \in M_i$. Then

$$\bar{r} \cdot \bar{m} \in \left(I/I^2\right)\left(M_i/M_{i+1}\right)$$

so equality holds. That is,

$$\left(I/I^2\right)\left(M_i/M_{i+1}\right) = \left(M_{i+1}/M_{i+2}\right)$$

for all $i \geq n$.

Thus, a generating set for M_i/M_{i+1} generates M_{i+1}/M_{i+2}, so the unions of generators of $M_0/M_1, \ldots, M_n/M_{n+1}$ generate $\operatorname{gr} M$. Since each M_i is f.g., $\operatorname{gr} M$ is as well. $\square$

We likely won't have any interesting homomorphisms $M \to \operatorname{gr} M$, but we do have a natural set map:

Def: Let $\mathcal{J}$ be the filtration $M = M_0 \supset M_1 \supset \ldots$, and $f \in M$. The initial form of f is

$$\operatorname{in}(f) := \begin{cases} 0 & \text{if } f \in \bigcap_{m=0}^{\infty} M_m \\ \bar{f} \in M_m/M_{m+1} \subseteq \operatorname{gr} M & \text{if } f \in M_m \setminus M_{m+1}. \end{cases}$$

Ex: Let $J = (xy + y^3, x^2) \subseteq R = k[x,y]$, and $I = (x,y)$. Consider $\operatorname{gr}_I R$.

Define $\operatorname{in}(J)$ to be the ideal in $\operatorname{gr}_I R$ generated by $\operatorname{in}(f)$ for all $f \in J$.

Note that $\operatorname{in}(x^2) = \overline{x^2} \in I^2/I^3$, and $\operatorname{in}(xy + y^3) = \overline{xy} \in I^2/I^3$.

However, $y^5 = \underbrace{(y^2 - x)(xy + y^3)}_{\substack{\shortparallel \\ xy^3 + y^5 - x^2y - xy^3}} + yx^2 \in J$

so $y^5 \in in(J)$, but y^5 is not generated by x^2 and xy in $gr_I R$. That is, $in(J)$ is not necessarily generated by the images of generators of J.

This construction gives us a way to turn an arbitrary Noetherian ring into a finitely generated algebra over a field:

Let $I \subseteq R$ be a max'l ideal, R Noetherian. Then

$$gr_I R = \underbrace{R/I}_{k} \oplus I/I^2 \oplus \cdots$$

and $I = (f_1, \ldots, f_n)$, so for $a \in I/I^2$, $a = r_1 f_1 + \cdots + r_n f_n$ where $r_i = 0$ or $r_i \notin I$.

If $a \in I^m/I^{m+1}$, $a = r_1 f_1 + \cdots + r_n f_n$, where each $r_i \in R \setminus I^m$ or $r_i = 0$. So by induction, each r_i is a polynomial in the f_i w/ coefficients in k.

This gives us a well-defined Hilbert function for local Noetherian rings:

Def: If R is a local ring w/ max'l ideal I, then the <u>Hilbert function of R</u> is

$$H_R(n) = \dim_{R/I} I^n/I^{n+1}.$$

If M is a f.g. R-module, define

$$H_M(n) = \dim_{R/I} I^nM/I^{n+1}M.$$

Note that these are just the Hilbert functions of $\mathrm{gr}_I R$ and $\mathrm{gr}_I M$, so we already know that for large values of n they agree with polynomials $P_R(n)$ and $P_M(n)$ of deg $\leq H_R(1) - 1$.

We can often learn about R by looking at $\mathrm{gr}_I R$. However, we need that no elements of R are lost in $\mathrm{gr}_I R$. i.e. we need $\bigcap_j I^j = 0$.

We'll soon see (via Krull intersection theorem) that this is usually the case.

Application: The tangent cone

Let $R = k[x_1, \dots, x_n]/J$ and $I = (x_1, \dots, x_n)$ s.t. $J \subset I$.
Let $X = V(J) \subseteq \mathbb{A}^n$. Then $\underset{(x_1, \dots, x_n)}{\overset{\shortparallel}{O}} \in X$.

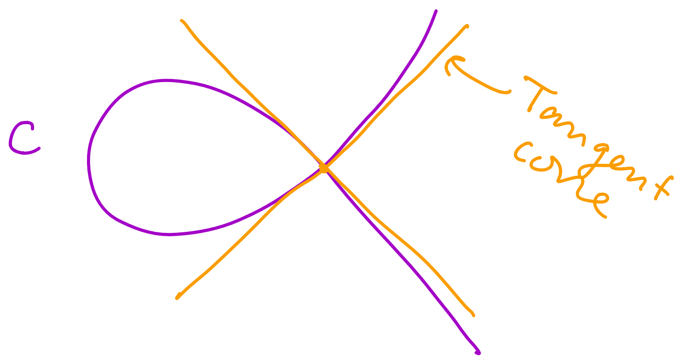
The <u>tangent cone</u> to X at O is

$$V(\mathrm{in}(J)) \subseteq \operatorname{Spec} k[x_1, \dots, x_n],$$

where $\mathrm{in}(J)$ is thought of as an ideal in $k[x_1, \dots, x_n]$. It consists of limits of secant lines through the origin.

Ex: $C = V(y^2 - x^2(x+1))$

$\mathrm{in}(y^2 - x^2(x+1))$
$= y^2 - x^2$
$= (y-x)(y+x)$



Ex: $C = V(y^2 - x^3)$

$\mathrm{in}(y^2 - x^3)$
$= y^2$